Study Material of B.Sc (Semester -II)
US02CMTH02 (Differential Equations-2)
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US02CMTH02
UNIT-3

1. Linear differential equations with constant coefficients

Definitions 1. A *linear differential equation* is an equation in which the dependent variable and its derivatives appear only in the first degree and are not multiplied together. Thus the general linear differential equation of the n^{th} order is of the form

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + P_n y = X, \tag{1}$$

where $P_1, P_2, \ldots, P_n$ and X are functions of x only. A linear equation of the form (1), with $P_1, P_2, \ldots, P_n$ constants, is called a *linear differential equation with constant coefficients.* The general form of such an equation is

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = X, \tag{2}$$

where $a_1, a_2, \ldots, a_n$ are constants and X is a function of x only.

Only linear differential equations with constant coefficients and homogeneous linear equations will be discussed in this chapter.

Theorem 2. *Let y_1 and y_2 be two solutions of a linear differential equation*

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = 0 \tag{3}$$

and C_1, C_2 be two arbitrary constants. Then $C_1 y_1 + C_2 y_2$ is also a solution of (3).

Proof. Since y_1 and y_2 are solutions of (3), we have,

$$\frac{d^n y_1}{dx^n} + a_1 \frac{d^{n-1} y_1}{dx^{n-1}} + \cdots + a_n y_1 = 0$$

and

$$\frac{d^n y_2}{dx^n} + a_1 \frac{d^{n-1} y_2}{dx^{n-1}} + \cdots + a_n y_2 = 0.$$

Thus we have,

$$\begin{aligned}
&\frac{d^n (C_1 y_1 + C_2 y_2)}{dx^n} + a_1 \frac{d^{n-1} (C_1 y_1 + C_2 y_2)}{dx^{n-1}} + \cdots + a_n (C_1 y_1 + C_2 y_2) \\
&\qquad = \frac{d^n (C_1 y_1)}{dx^n} + a_1 \frac{d^{n-1} (C_1 y_1)}{dx^{n-1}} + \cdots + a_n C_1 y_1 \\
&\qquad\quad + \frac{d^n (C_2 y_2)}{dx^n} + a_1 \frac{d^{n-1} (C_2 y_2)}{dx^{n-1}} + \cdots + a_n C_2 y_2 && (4) \\
&\qquad = C_1 \left[\frac{d^n y_1}{dx^n} + a_1 \frac{d^{n-1} y_1}{dx^{n-1}} + \cdots + a_n y_1 \right] \\
&\qquad\quad + C_2 \left[\frac{d^n y_2}{dx^n} + a_1 \frac{d^{n-1} y_2}{dx^{n-1}} + \cdots + a_n y_2 \right] && (5) \\
&\qquad = 0.
\end{aligned}$$

This proves the theorem. □

Definitions 3. For linearly independent solutions $y_1, y_2, \ldots, y_n$ of (3), $u = C_1y_1 + C_2y_2 + \cdots + C_ny_n$ is called the *general solution* of (3), where $C_1, C_2, \ldots, C_n$ are arbitrary constants. If a function v of x, when substituted for y, satisfies (2), $y = v$ is called a *particular solution* of (2). In this case,

$$\frac{d^n v}{dx^n} + a_1 \frac{d^{n-1} v}{dx^{n-1}} + \cdots + a_n v = X.$$

Let $y = u$ be a general solution of (3) and $y = v$ be a particular solution of (2). Then,

$$\begin{aligned}
&\frac{d^n(u+v)}{dx^n} + a_1 \frac{d^{n-1}(u+v)}{dx^{n-1}} + \cdots + a_n(u+v) \\
&= \left(\frac{d^n u}{dx^n} + a_1 \frac{d^{n-1} u}{dx^{n-1}} + \cdots + a_n u\right) \\
&\qquad + \left(\frac{d^n v}{dx^n} + a_1 \frac{d^{n-1} v}{dx^{n-1}} + \cdots + a_n v\right) \\
&= 0 + X \qquad \text{(because } u \text{ is a solution of (3))} \\
&= X.
\end{aligned}$$

This shows that $y = u + v$ is the general solution of (2).
The part $u = C_1y_1 + C_2y_2 + \cdots + C_ny_n$ is called the *complementary function* (C.F.) and the part v is called the *particular integral* (P.I.) of (2). Thus in order to solve the differential equation (2), we have to first find C.F., *i.e.*, the complementary solution of (3) and then P.I., *i.e.*, a particular solution of (2).

4. Operators: The symbols $D, D^2, \ldots, D^n$ are used generally for the operators $\frac{d}{dx}, \frac{d^2}{dx^2}, \ldots, \frac{d^n}{dx^n}$ respectively. The equation (2) can be written in the symbolic form as

$$(D^n + a_1 D^{n-1} + \cdots + a_n)y = X, \ \textit{i.e.}, \ f(D)y = X,$$

where $f(D) = D^n + a_1D^{n-1} + \cdots + a_n$. $f(D)$ can be factorized by ordinary rules of algebra and the factors may be taken in any order. For example,

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = (D^2 + 2D - 3)y = (D-1)(D+3)y = (D+3)(D-1)y.$$

2. Rules to find complementary function

In order to find the complementary function for the equation (3), we show that each root of the polynomial equation $f(D) = 0$ gives rise to a solution of the differential equation (3). Further, all these solutions are linearly independent. We show in the theorem (6) how is this achieved. However, before this we assume the following result for a polynomial.

Proposition 5. *Consider the polynomial equation*

$$p(x) = x^n + a_1x^{n-1} + \cdots + a_n,$$

with $a_i \in \mathbb{R}$. Then there are precisely n complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$, not necessarily distinct, such that the following holds.

$$p(x) = (x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_n).$$

Further, the complex numbers in the above expression occur in pair with their conjugates.

Theorem 6. *The differential equation*

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = 0, \tag{6}$$

with constant coefficients always admits the general solution.

Proof. We write the equation (6) as

$$f(D)y = (D^n + a_1 D^{n-1} + \ldots + a_n)y = 0. \tag{7}$$

To solve this equation, we consider the polynomial equation

$$f(D) = (D^n + a_1 D^{n-1} + \ldots + a_n) = 0. \tag{8}$$

This polynomial equation is called the *auxiliary equation* (A.E.) for the given differential equation (6). Treating D as a variable, and using the Proposition 5, this polynomial equation has precisely n roots, say $\alpha_1, \alpha_2, \ldots, \alpha_n$. Hence in this case the (7) becomes

$$(D - \alpha_1)(D - \alpha_2) \cdots (D - \alpha_n)y = 0. \tag{9}$$

Clearly, for all $j = 1, 2, \ldots, n$, the solutions of the equations

$$(D - \alpha_j)y = 0 \tag{10}$$

are also solutions of (6). As a result, we need only to solve (10) and get n solutions of (6). Note that for a fixed j, (10) gives,

$$\begin{aligned}
&\frac{dy}{dx} - \alpha_j y = 0\\
\Rightarrow &\frac{dy}{y} = \alpha_j dx\\
\Rightarrow &\log y = \alpha_j x + F_j \quad (F_j \text{ is constant for each } j.)\\
\Rightarrow &y = e^{\alpha_j x + F_j} = e^{\alpha_j x} e^{F_j} = K_j e^{\alpha_j x},
\end{aligned}$$

where K_j is a constant. We denote this solution by $y_j = K_j e^{\alpha_j x}$.

Case I. All the roots of (8) are real and distinct.

In this case the general solution of (6) is given by

$$y = H_1 y_1 + H_2 y_2 + \cdots + H_n y_n,$$

with constants $H_1, H_2, \ldots, H_n$. Taking $C_j = H_j K_j$ the general solution of (6) is

$$y = C_1 e^{\alpha_1 x} + C_2 e^{\alpha_2 x} + \cdots + C_n e^{\alpha_n x},$$

where $C_1, C_2, \ldots, C_n$ are arbitrary constants.

Case II. Two roots of (8) are equal and rest of its roots are real and distinct.

Without loss of generality, we assume that $\alpha_1 = \alpha_2$. Then (9) will be satisfied by the solution of $(D - \alpha_1)^2 y = 0$. Let $(D - \alpha_1)y = z$. Then $(D - \alpha_1)z = 0$, *i.e.*, $\frac{dz}{z} = \alpha_1 dx$, which gives $z = C_1 e^{\alpha_1 x}$. Thus we have $(D - \alpha_1)y = z = C_1 e^{\alpha_1 x}$ or $\frac{dy}{dx} - \alpha_1 y = C_1 e^{\alpha_1 x}$, which is a linear equation with integrating factor $e^{\int -\alpha_1 dx} = e^{-\alpha_1 x}$ and hence its solution is

$$ye^{-\alpha_1 x} = \int C_1 e^{\alpha_1 x} e^{-\alpha_1 x} dx + C_2 = C_1 x + C_2.$$

Thus the general solution of (6) is

$$y = (C_1 x + C_2)e^{\alpha_1 x} + C_3 e^{\alpha_3 x} + \cdots + C_n e^{\alpha_n x},$$

where $C_1, C_2, \ldots, C_n$ are arbitrary constants. Changing the indices of the constants and rewriting conveniently, the general solution of (6) becomes

$$y = (C_1 + C_2 x)e^{\alpha_1 x} + C_3 e^{\alpha_3 x} + \cdots + C_n e^{\alpha_n x},$$

In general, we can show similarly that if only r roots of (8) are equal, say, $\alpha_1 = \alpha_2 = \cdots = \alpha_r$ and the rest are real and distinct, then the general solution of (6) is

$$y = (C_1 + C_2 x + \cdots + C_r x^{r-1})e^{\alpha_1 x} + C_{r+1} e^{\alpha_{r+1} x} + \cdots + C_n e^{\alpha_n x},$$

where $C_1, C_2, \ldots, C_n$ are arbitrary constants. This method can be generalized when $\alpha_1 = \alpha_2 = \cdots = \alpha_r, \alpha_{r+1} = \alpha_{r+2} = \cdots = \alpha_p\ , \alpha_{p+1} = \alpha_{p+2} = \cdots = \alpha_q, \ldots,$ $\alpha_{s+1} = \alpha_{s+2} = \cdots = \alpha_t$ and the rest of the roots are real and distinct.

Case III. One pair of roots of (8) is imaginary, say $\alpha_1 = \alpha + i\beta, \alpha_2 = \alpha - i\beta$ and the rest are real and distinct. Then the general solution of (6) is

$$\begin{aligned} y &= C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x} + C_3 e^{\alpha_3 x} + \cdots + C_n e^{\alpha_n x} \\ &= e^{\alpha x}(C_1 e^{i\beta x} + C_2 e^{-i\beta x}) + C_3 e^{\alpha_3 x} + \cdots + C_n e^{\alpha_n x} \\ &= e^{\alpha x}[C_1(\cos\beta x + i\sin\beta x) + C_2(\cos\beta x - i\sin\beta x)] \\ &\quad + C_3 e^{\alpha_3 x} + \cdots + C_n e^{\alpha_n x} \quad (\text{because } e^{i\theta} = \cos\theta + i\sin\theta) \\ &= e^{\alpha x}(P_1\cos\beta x + P_2\sin\beta x) + C_3 e^{\alpha_3 x} + \cdots + C_n e^{\alpha_n x}, \end{aligned}$$

where $P_1 = C_1 + C_2$ and $P_2 = i(C_1 - C_2)$.

Case IV. Two pairs of roots of (8) are imaginary and equal, say, $\alpha_1 = \alpha + i\beta = \alpha_2$ and $\alpha_3 = \alpha - i\beta = \alpha_4$ and the rest are real and distinct. Then as in the previous cases, the general solution of (6) is

$$y = e^{\alpha x}[(C_1 + C_2 x)\cos\beta x + (C_3 + C_4 x)\sin\beta x] + C_5 e^{\alpha_5 x} + \cdots + C_n e^{\alpha_n x}.$$

Thus in all combinations of these cases, the general solution of (6) exists. □

Example 7. Solve

$$\frac{d^4y}{dx^4} - 2\frac{d^3y}{dx^3} + 5\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 4y = 0. \tag{11}$$

Solution. Here we have $(D^4 - 2D^3 + 5D^2 - 8D + 4)y = 0$. So, as in (8), we find the roots of the auxiliary equation (A.E.) $D^4 - 2D^3 + 5D^2 - 8D + 4 = 0$. To this end,

$$\begin{aligned} D^4 - 2D^3 + 5D^2 - 8D + 4 &= (D-1)(D^3 - D^2 + 4D - 4) \\ &= (D-1)(D^2+4)(D-1) \\ &= (D-1)^2(D-2i)(D+2i) = 0. \end{aligned}$$

The roots of the above equation are $1, 1, \pm 2i$. That is, root 1 is repeated twice, the other two roots $2i$ and $-2i$ are in the pair of complex conjugates. Hence the general solution of (11) is

$$\begin{aligned} y &= (C_1 + C_2 x)e^x + e^{0x}(C_3\cos 2x + C_4\sin 2x) \\ &= (C_1 + C_2 x)e^x + C_3\cos 2x + C_4\sin 2x. \end{aligned}$$

□

Example 8. Solve $(D^3 + 6D^2 + 12D + 8)y = 0$.

Solution. We can write $D^3 + 6D^2 + 12D + 8 = 0$ as $(D+2)^3 = 0$. Hence the roots of this polynomial are $-2, -2, -2$, that is, the root -2 is repeated thrice. Hence the general solution is $y = (C_1 + C_2 x + C_3 x^2)e^{-2x}$. □

Example 9. Solve $(D^2 - 5D + 4)y = 0$.

Solution. We can write $(D^2 - 5D + 4)y = 0$ as $(D-4)(D-1) = 0$. Hence the roots of this polynomial are $4, 1$. Hence the general solution is $y = C_1 e^{4x} + C_2 e^x$. □

Example 10. Solve $(D^3 - 4D^2 + 5D - 2)y = 0$.

3. Rules for finding particular integral

The complementary function is a solution of the differential equation (3). In this section we deal with the general solution of (2), *i.e.*, the sum of P.I. and C.F. of (2).

Definition 11. If X is a function of x, then $\frac{1}{f(D)}X$ stands for a function Z of x such that $f(D)Z = X$. This Z is called the *particular solution* of $f(D)y = X$, *i.e.*, we write P.I. $= \frac{1}{f(D)}X$. Obviously $f(D)$ and $\frac{1}{f(D)}$ are the inverses of each other. In particular, $\frac{1}{D}X$ stands for $\int X dx$. We note that P.I. is always free from constant.

Our main concern in this section is to obtain the particular solutions of $f(D)y = X$. We start with the simple form $\mathbf{f(D)} = \mathbf{D} - \alpha$. Now,

$$\frac{1}{f(D)}X = \frac{1}{D-\alpha}X = y. \tag{12}$$

Operating $D - \alpha$ on (12), we get,

$$(D-\alpha)\frac{1}{D-\alpha}X = (D-\alpha)y \Rightarrow X = (D-\alpha)y = \frac{dy}{dx} - \alpha y$$
$$\Rightarrow \frac{dy}{dx} - \alpha y = X,$$

which is a linear equation with integrating factor $e^{\int -\alpha dx} = e^{-\alpha x}$, and its solution is $ye^{-\alpha x} = \int Xe^{-\alpha x}dx$. As a result, the particular integral is given by $\frac{1}{D-\alpha}X = e^{\alpha x}\int Xe^{-\alpha x}dx$.

Example 12. Solve $(D^2 + a^2)y = \operatorname{cosec} ax$.

Solution. Here the auxiliary equation is

$$D^2 + a^2 = 0 \quad i.e., \ D = \pm ai.$$

Hence C.F. $= C_1 \cos ax + C_2 \sin ax$. Now,

$$\text{P.I.} \ = \frac{1}{D^2+a^2}\operatorname{cosec} ax = \frac{1}{2ia}\left[\frac{1}{D-ia} - \frac{1}{D+ia}\right]\operatorname{cosec} ax, \tag{13}$$

where

$$\begin{aligned}\frac{1}{D-ia}\operatorname{cosec} ax &= e^{iax}\int e^{-iax}\operatorname{cosec} ax\, dx \\ &= e^{iax}\int(\cos ax - i\sin ax)\frac{1}{\sin ax}\,dx \\ &= e^{iax}\int(\cot ax - i)\,dx \\ &= e^{iax}\left(\frac{1}{a}\log(\sin ax) - ix\right).\end{aligned}$$

Similarly,

$$\frac{1}{D+ia}\operatorname{cosec} ax = e^{-iax}\left(\frac{1}{a}\log(\sin ax) + ix\right).$$

Putting both these values in (13) we get,

$$\begin{aligned}\text{P.I.} &= \frac{1}{2ia}\left[e^{iax}\left(\frac{1}{a}\log(\sin ax) - ix\right) - e^{-iax}\left(\frac{1}{a}\log(\sin ax) + ix\right)\right] \\ &= \frac{1}{a^2}\log(\sin ax)\left(\frac{e^{iax} - e^{-iax}}{2i}\right) - \frac{x}{a}\left(\frac{e^{iax} + e^{-iax}}{2}\right) \\ &= \frac{1}{a^2}\sin ax\log(\sin ax) - \frac{x}{a}\cos ax.\end{aligned}$$

Hence the general solution is

$$y = \text{C.F.} + \text{P.I.} = C_1\cos ax + C_2\sin ax + \frac{1}{a^2}\sin ax\log(\sin ax) - \frac{x}{a}\cos ax.$$

□

Example 13. Solve $(D^2 + 4)y = \sec ax$.

Now we describe the rules to find the particular integral of (2) for different types of X.

Rules for finding the particular integral when $\mathbf{X} = e^{\mathbf{mx}}$ with a constant m.

Theorem 14. *Obtain rule for finding the particular integral of $f(D)y = e^{mx}$ where m is constant.*

Proof. For $f(D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n$, we have to find P.I. of $f(D)y = e^{mx}$. We know that

$$D(e^{mx}) = me^{mx};\ D^2(e^{mx}) = m^2 e^{mx};\ D^3(e^{mx}) = m^3 e^{mx};\ D^n(e^{mx}) = m^n e^{mx}.$$

Therefore,

$$\begin{aligned} f(D)e^{mx} &= (D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n)e^{mx} \\ &= D^n e^{mx} + a_1 D^{n-1} e^{mx} + a_2 D^{n-2} e^{mx} + \cdots + a_n e^{mx} \\ &= m^n e^{mx} + a_1 m^{n-1} e^{mx} + \cdots + a_n e^{mx} \\ &= (m^n + a_1 m^{n-1} + \cdots + a_n) e^{mx} \\ &= f(m) e^{mx}. \end{aligned}$$

Thus $f(D)e^{mx} = f(m)e^{mx}$. Operating $\frac{1}{f(D)}$ on both sides we get,

$$e^{mx} = \frac{1}{f(D)} f(m) e^{mx} = f(m) \frac{1}{f(D)} e^{mx}. \tag{14}$$

Suppose $f(m) \neq 0$. Dividing (14) by $f(m)$, we get,

$$\frac{1}{f(m)} e^{mx} = \frac{1}{f(D)} e^{mx}.$$

Thus P.I. is given by

$$\frac{1}{f(D)} e^{mx} = \frac{1}{f(m)} e^{mx}, \tag{15}$$

provided $f(m) \neq 0$.
Now if $f(m) = 0$, the above rule fails. So, we proceed further in the following way. Since $f(m) = 0$, m is a root of $f(D) = 0$, *i.e.*, $(D - m)$ is a factor of $f(D)$. Let r be the largest integer such that $(D - m)^r$ is a factor of $f(D)$. In other words, m is a root of $f(D) = 0$ with multiplicity r. In this case, $f(D) = (D - m)^r \varphi(D)$, with $\varphi(m) \neq 0$. Hence,

$$\begin{aligned} \frac{1}{f(D)} e^{mx} &= \frac{1}{(D-m)^r \varphi(D)} e^{mx} \\ &= \frac{1}{(D-m)^r} \frac{1}{\varphi(D)} e^{mx} \\ &= \frac{1}{(D-m)^r} \frac{1}{\varphi(m)} e^{mx} \quad \text{(by (15))} \\ &= \frac{1}{\varphi(m)} \frac{1}{(D-m)^r} e^{mx}. \end{aligned} \tag{16}$$

Now,

$$\begin{aligned} \frac{1}{(D-m)} e^{mx} &= e^{mx} \int e^{-mx} e^{mx} dx = x e^{mx}; \\ \frac{1}{(D-m)^2} e^{mx} &= \frac{1}{(D-m)} x e^{mx} = e^{mx} \int e^{-mx} x e^{mx} dx = \frac{x^2}{2} e^{mx}. \end{aligned}$$

In general,

$$\frac{1}{(D-m)^r} e^{mx} = \frac{x^r}{r!} e^{mx}.$$

Putting this value in (16), we get, P.I. $= \frac{1}{f(D)} e^{mx} = \frac{x^r}{\varphi(m) r!} e^{mx}$,
where $f(D) = (D - m)^r \varphi(D)$, and $\varphi(m) \neq 0$. □

Remark 15. In general we can solve the above equation when e is replaced by any positive number a. In this case, the equation under consideration becomes

$$f(D)y = a^{mx}. \quad \text{That is,} \quad f(D)y = e^{(\log a)mx}.$$

and its P.I. becomes

$$\frac{1}{f(m\log a)} a^{mx}.$$

Example 16. Solve

(1) $(D^2 - 3D + 5)y = e^{-x}$.

(2) $(D^3 - 5D^2 + 7D - 3)y = \cosh x$.

(3) $(D^3 - 1)y = (e^x - 1)^2$.

Solution. (1) Here the auxiliary equation is

$$D^2 - 3D + 5 = 0 \Rightarrow D = \frac{3 \pm \sqrt{9-20}}{2} = \frac{3}{2} \pm i\frac{\sqrt{11}}{2}.$$

Thus the C.F. $= e^{3x/2}\left(C_1 \cos\left(\sqrt{\frac{11}{2}}\right) x + C_2 \sin\left(\sqrt{\frac{11}{2}}\right) x\right)$. Now,

$$\text{P.I.} \; = \frac{1}{D^2 - 3D + 5} e^{-x} = \frac{1}{(-1)^2 - 3(-1) + 5} e^{-x} = \frac{1}{9} e^{-x}.$$

Hence the general solution is

$$y = \text{C.F.} + \text{P.I.} = e^{3x/2}\left(C_1 \cos\left(\sqrt{\tfrac{11}{2}}\right) x + C_2 \sin\left(\sqrt{\tfrac{11}{2}}\right) x\right) + \frac{1}{9} e^{-x}.$$

(2) Here the auxiliary equation is

$$\begin{aligned}
&D^3 - 5D^2 + 7D - 3 = 0 \\
\Rightarrow &D^2(D-1) - 4D(D-1) + 3(D-1) = 0 \\
\Rightarrow &(D-1)(D^2 - 4D + 3) = 0 \\
\Rightarrow &(D-1)^2(D-3) = 0 \\
\Rightarrow &D = 1, 1, 3.
\end{aligned}$$

Thus the C.F. $= (C_1 + C_2 x)e^x + C_3 e^{3x}$. Now

$$\begin{aligned}
\text{P.I.} \; &= \frac{1}{D^3 - 5D^2 + 7D - 3} \cosh x \\
&= \frac{1}{(D-1)^2(D-3)} \left(\frac{e^x + e^{-x}}{2}\right) \\
&= \frac{1}{2(D-1)^2(D-3)} e^x + \frac{1}{2(D-1)^2(D-3)} e^{-x} \\
&= \frac{1}{2(D-1)^2(1-3)} e^x + \frac{1}{2(-1-1)^2(-1-3)} e^{-x} \\
&= -\frac{1}{4} \frac{1}{(D-1)^2} e^x - \frac{1}{32} e^{-x} \\
&= -\frac{1}{4} \frac{x^2}{2!} e^x - \frac{1}{32} e^{-x} \\
&= -\frac{x^2}{8} e^x - \frac{1}{32} e^{-x}.
\end{aligned}$$

Hence the general solution is

$$y = (C_1 + C_2 x)e^x + C_3 e^{3x} - \frac{x^2}{8} e^x - \frac{1}{32} e^{-x}.$$

(3) Here the auxiliary equation is

$$(D^3-1)=0 \Rightarrow (D-1)(D^2+D+1)=0 \Rightarrow D=1,-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}.$$

Thus the C.F. $= C_1e^x + e^{-x/2}\left[C_2\cos\left(\frac{\sqrt{3}}{2}x\right)+C_3\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$. Now

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^3-1}(e^x-1)^2 \\
&= \frac{1}{D^3-1}(e^{2x}-2e^x+1) \\
&= \frac{1}{D^3-1}e^{2x} - \frac{1}{D^3-1}2e^x + \frac{1}{D^3-1}e^{0x} \\
&= \frac{1}{7}e^{2x} - \frac{1}{(D-1)(D^2+D+1)}2e^x - 1e^{0x} \\
&= \frac{1}{7}e^{2x} - \frac{2}{3}\frac{1}{(D-1)}e^x - 1 \\
&= \frac{e^{2x}}{7} - \frac{2}{3}\frac{x}{1!}e^x - 1 \\
&= \frac{e^{2x}}{7} - \frac{2xe^x}{3} - 1.
\end{aligned}$$

Hence the general solution is

$$y = C_1e^x + e^{-x/2}\left[C_2\cos\left(\frac{\sqrt{3}}{2}x\right)+C_3\sin\left(\frac{\sqrt{3}}{2}x\right)\right] + \frac{e^{2x}}{7} - \frac{2xe^x}{3} - 1.$$

□

Example 17. Solve $(D^2-5D+6)y = 4e^x$ subject to the conditions that y(0) = y'(0) = 1 . Hence find y(16) .

Rules for finding the particular integral when X = sin mx or cos mx with a constant m.

Theorem 18. *Obtain rule for finding the particular integral of $f(D)y = sinmx$ where m is constant.*

Proof. To find P.I. of $f(D)y = \sin mx$ we have to consider following cases:

Case I. $f(D)$ contains only even powers of D, so that it is a function of D^2. Hence $f(D) = \varphi(D^2)$. Suppose now that $\varphi(-m^2) \neq 0$. Then

$$\text{P.I.} = \frac{1}{f(D)} \sin mx = \frac{1}{\varphi(D^2)} \sin mx = \frac{1}{\varphi(-m^2)} \sin mx.$$

And if $\varphi(-m^2) = 0$, then

$$\text{P.I.} = \frac{1}{f(D)} \sin mx = \frac{1}{\varphi(D^2)} \sin mx = \frac{-x}{2m\psi(-m^2)} \cos mx,$$

where $f(D) = \varphi(D^2) = (D^2 + m^2)\psi(D^2)$, $\psi(-m^2) \neq 0$.

Case II. Again we consider the case when $f(D)$ contains a mixture of odd as well as even powers of D. We break up $f(D)$ into its even and odd parts to get,

$$f(D) = \varphi_1(D^2) + D\varphi_2(D^2).$$

In this case,

$$\text{P.I.} = \frac{1}{f(D)} \sin mx = \frac{P \sin mx - mQ \cos mx}{P^2 + m^2 Q^2},$$

where $P = \varphi_1(-m^2)$ and $Q = \varphi_2(-m^2)$. □

Remark 19. Similarly for finding P.I. of $f(D)y = \cos mx$ we have to consider following cases:

Case I. $f(D)$ contains only even powers of D, so that it is a function of D^2. Hence $f(D) = \varphi(D^2)$. Suppose now that $\varphi(-m^2) \neq 0$. Then

$$\text{P.I.} = \frac{1}{f(D)} \cos mx = \frac{1}{\varphi(D^2)} \cos mx = \frac{1}{\varphi(-m^2)} \cos mx.$$

And if $\varphi(-m^2) = 0$, then

$$\text{P.I.} = \frac{1}{f(D)} \cos mx = \frac{1}{\varphi(D^2)} \cos mx = \frac{x}{2m\psi(-m^2)} \sin mx,$$

where $f(D) = \varphi(D^2) = (D^2 + m^2)\psi(D^2)$, $\psi(-m^2) \neq 0$.

Case II. Again we consider the case when $f(D)$ contains a mixture of odd as well as even powers of D. We break up $f(D)$ into its even and odd parts to get,

$$f(D) = \varphi_1(D^2) + D\varphi_2(D^2).$$

In this case,

$$\text{P.I.} = \frac{1}{f(D)} \cos mx = \frac{P \cos mx + mQ \sin mx}{P^2 + m^2 Q^2},$$

where $P = \varphi_1(-m^2)$ and $Q = \varphi_2(-m^2)$.

Notes: One can easily prove the following results. We shall need them.

(1) $\frac{1}{D^2+m^2} \cos mx = \frac{x}{2m} \sin mx$.

(2) $\frac{1}{D^2+m^2} \sin mx = -\frac{x}{2m} \cos mx$.

Example 20. Solve

(1) $(D^2 + 3D + 2)y = \cos 3x$.

(2) $(D + 1)^2 y = (1 + \sin x)^2$.

Solution. (1) Here the auxiliary equation is

$$D^2 + 3D + 2 = 0 \Rightarrow (D+2)(D+1) = 0 \Rightarrow D = -1, -2.$$

Thus the C.F. $= C_1 e^{-x} + C_2 e^{-2x}$. Now,

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2+3D+2}\cos 3x \\
&= \frac{1}{-9+3D+2}\cos 3x \\
&= \frac{1}{3D-7}\cos 3x = (3D+7)\frac{1}{(3D-7)(3D+7)}\cos 3x \\
&= (3D+7)\frac{1}{9D^2-49}\cos 3x = -\frac{1}{130}(3D+7)\cos 3x \\
&= -\frac{1}{130}(3D(\cos 3x) + 7\cos 3x) \\
&= -\frac{1}{130}(-9\sin 3x + 7\cos 3x).
\end{aligned}$$

Hence the general solution is

$$\begin{aligned}
y &= \text{C.F.} + \text{P.I.} \\
&= C_1 e^{-x} + C_2 e^{-2x} - \frac{1}{130}(-9\sin 3x + 7\cos 3x).
\end{aligned}$$

(2) Here the auxiliary equation is

$$(D+1)^2 = 0 \Rightarrow D = -1, -1.$$

Thus the C.F. $= (C_1 + C_2 x)e^{-x}$. Now,

$$\begin{aligned}
\text{P.I.} &= \frac{1}{(D+1)^2}(1 + 2\sin x + \sin^2 x) \\
&= \frac{1}{(D+1)^2}\left(1 + 2\sin x + \frac{1-\cos 2x}{2}\right) \\
&= \frac{1}{(D+1)^2}\frac{3}{2} + 2\frac{1}{(D+1)^2}\sin x - \frac{1}{(D+1)^2}\frac{\cos 2x}{2} \\
&= \frac{3}{2}\frac{1}{(D+1)^2}e^{0x} + 2\frac{1}{(D^2+2D+1)}\sin x - \frac{1}{2}\frac{1}{(D^2+2D+1)}\cos 2x \\
&= \frac{3}{2} + 2\frac{1}{2D}\sin x - \frac{1}{2}\frac{1}{(2D-3)}\cos 2x \\
&= \frac{3}{2} - \cos x - \frac{1}{2}(2D+3)\frac{1}{(4D^2-9)}\cos 2x \\
&= \frac{3}{2} - \cos x + \frac{1}{50}(2D+3)\cos 2x \\
&= \frac{3}{2} - \cos x + \frac{1}{50}(-4\sin 2x + 3\cos 2x).
\end{aligned}$$

Hence the general solution is

$$\begin{aligned}
y &= \text{C.F.} + \text{P.I.} \\
&= (C_1 + C_2 x)e^{-x} + \frac{3}{2} - \cos x + \frac{1}{50}(-4\sin 2x + 3\cos 2x).
\end{aligned}$$

□

21. Rule for finding the particular integral when X is of the form x^m with a constant positive integer m.

Here we have to evaluate $\frac{1}{f(D)}x^m$. First we take out the lowest degree term – with sign – from $f(D)$, leaving the remaining factor of the form $1 \pm \varphi(D)$. We shall expand this factor into a series of powers of D. In fact, since $D^k x^m = 0$ for each $k > m$, the power series will reduce to a finite polynomial in D with possible highest power m. Let us recall certain power series to be used.

$$\begin{aligned}
(1-D)^{-1} &= 1 + D + D^2 + D^3 + \cdots \\
(1-D)^{-2} &= 1 + 2D + 3D^2 + 4D^3 + \cdots \\
(1-D)^{-3} &= 1 + 3D + 6D^2 + 10D^3 + \cdots + \frac{(r+1)(r+2)}{2}D^r + \cdots \\
(1+D)^{-1} &= 1 - D + D^2 - D^3 + \cdots \\
(1+D)^{-2} &= 1 - 2D + 3D^2 - 4D^3 + \cdots \\
(1+D)^{-3} &= 1 - 3D + 6D^2 - 10D^3 \\
&\qquad + \cdots + (-1)^r \frac{(r+1)(r+2)}{2}D^r + \cdots
\end{aligned}$$

In general, using Binomial Theorem, for $n \in \mathbb{Z}$,

$$(1-D)^n = 1 + nD + \frac{n(n-1)}{2!}D^2 + \frac{n(n-1)(n-1)}{3!}D^3 + \cdots$$

Example 22. Solve

(1) $(D^3 - D^2 - 6D)y = x^3$.

(2) $(D^3 + 4D)y = \sin 2x + 3e^{2x} + 2x$.

Solution. (1) Here the auxiliary equation is

$$\begin{aligned}
&D^3 - D^2 - 6D = 0 \\
&\Rightarrow D(D^2 - D - 6) = 0 \\
&\Rightarrow D(D-3)(D+2) = 0 \\
&\Rightarrow D = 0, 3, -2.
\end{aligned}$$

Thus C.F. $= C_1 + C_2e^{3x} + C_3e^{-2x}$. Now,

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^3 - D^2 - 6D}x^3 \\
&= \frac{1}{-6D(1 - \frac{D^2-D}{6})}x^3 \\
&= -\frac{1}{6D}\left(1 - \frac{D^2-D}{6}\right)^{-1}x^3 \\
&= -\frac{1}{6D}\left[1 + \frac{D^2-D}{6} + \left(\frac{D^2-D}{6}\right)^2 + \left(\frac{D^2-D}{6}\right)^3 + \cdots\right]x^3 \\
&= -\frac{1}{6D}\left[x^3 + \frac{D^2-D}{6}x^3 + \left(\frac{D^4 - 2D^3 + D^2}{36}\right)x^3 - \frac{D^3}{216}x^3 + \cdots\right] \\
&= -\frac{1}{6D}\left[x^3 + \frac{6x - 3x^2}{6} + \frac{0 - 12 + 6x}{36} - \frac{1}{36}\right] \\
&= -\frac{x^4}{24} - \frac{x^2}{12} + \frac{x^3}{36} + \frac{x}{18} - \frac{x^2}{72} + \frac{x}{216} \\
&= -\frac{x^4}{24} + \frac{x^3}{36} - \frac{7x^2}{72} + \frac{13x}{216}.
\end{aligned}$$

Hence the general solution is

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} \\ &= C_1 + C_2e^{3x} + C_3e^{-2x} - \frac{x^4}{24} + \frac{x^3}{36} - \frac{7x^2}{72} + \frac{13x}{216}. \end{aligned}$$

(2) Here the auxiliary equation is

$$D^3 + 4D = 0 \Rightarrow D(D^2+4) = 0 \Rightarrow D = 0, \pm 2i.$$

Thus the

$$\begin{aligned} \text{C.F.} &= C_1e^{0x} + C_2\cos 2x + C_3 \sin 2x \\ &= C_1 + C_2 \cos 2x + C_3 \sin 2x. \end{aligned}$$

Now,

$$\begin{aligned} \text{P.I.} &= \frac{1}{(D^2+4)D}(\sin 2x + 3e^{2x} + 2x) \\ &= \frac{1}{D^2+4}\left(-\frac{\cos 2x}{2} + \frac{3e^{2x}}{2} + x^2\right) \\ &= -\frac{1}{2}\frac{1}{D^2+4}\cos 2x + \frac{3}{2}\frac{1}{D^2+4}e^{2x} + \frac{1}{D^2+4}x^2 \\ &= -\frac{x}{8}\sin 2x + \frac{3}{16}e^{2x} + \frac{1}{4}\frac{1}{(1+\frac{D^2}{4})}x^2 \\ &= -\frac{x}{8}\sin 2x + \frac{3}{16}e^{2x} + \frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1}x^2 \\ &= -\frac{x}{8}\sin 2x + \frac{3}{16}e^{2x} + \frac{1}{4}\left(1 - \frac{D^2}{4} + \left(\frac{D^2}{4}\right)^2 - \dots\right)x^2 \\ &= -\frac{x}{8}\sin 2x + \frac{3}{16}e^{2x} + \frac{1}{4}\left(x^2 - \frac{D^2}{4}x^2 + \frac{D^4}{16}x^2 - \dots\right) \\ &= -\frac{x}{8}\sin 2x + \frac{3}{16}e^{2x} + \frac{x^2}{4} - \frac{1}{8}. \end{aligned}$$

Hence the general solution is

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} \\ &= C_1 + C_2\cos 2x + C_3 \sin 2x - \frac{x}{8}\sin 2x + \frac{3}{16}e^{2x} + \frac{x^2}{4} - \frac{1}{8}. \end{aligned}$$

□

Rule for finding the particular integral when $\mathbf{X = e^{ax}V}$ where V is a function of x.

Theorem 23. *In usual notation prove that* $\frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V$, *where* V *is a function of* x.

Proof. Here we have to evaluate $\frac{1}{f(D)}e^{ax}V$. Let us note that for any function W of x,

$$\begin{aligned} D(e^{ax}W) &= e^{ax}DW + ae^{ax}W = e^{ax}(D+a)W. \\ D^2(e^{ax}W) &= D(e^{ax}(D+a)W) = e^{ax}(D+a)^2W. \end{aligned}$$

In general,

$$D^n(e^{ax}W) = e^{ax}(D+a)^nW.$$

Therefore,

$$\begin{aligned} f(D)e^{ax}W &= (D^n + a_1 D^{n-1} + \cdots + a_n)e^{ax}W \\ &= e^{ax}[(D+a)^n + a_1(D+a)^{n-1} + \cdots + a_n]W \\ &= e^{ax} f(D+a)W. \end{aligned}$$

Suppose now that W is given by $f(D+a)W = V$. Then $W = \frac{1}{f(D+a)}V$. As a result, $f(D)e^{ax}\frac{1}{f(D+a)}V = e^{ax}V$. So, operating $\frac{1}{f(D)}$ on both the sides gives,

$$e^{ax}\frac{1}{f(D+a)}V = \frac{1}{f(D)}e^{ax}V.$$

Hence we have $\frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V$, where V is a function of x. □

Example 24. Solve $(D^2+2)y = (x^2+1)e^{3x} + e^x \cos 2x$.

Solution. Here the auxiliary equation is

$$D^2 + 2 = 0 \Rightarrow D = \pm i\sqrt{2}.$$

Thus the C.F. $= C_1\cos(\sqrt{2}\,x) + C_2\sin(\sqrt{2}\,x)$. Now

$$\begin{aligned} \text{P.I. } &= \frac{1}{D^2+2}((x^2+1)e^{3x} + e^x\cos 2x) \\ &= \frac{1}{D^2+2}e^{3x}(x^2+1) + \frac{1}{D^2+2}e^x\cos 2x \\ &= e^{3x}\frac{1}{(D+3)^2+2}(x^2+1) + e^x\frac{1}{(D+1)^2+2}\cos 2x \\ &= e^{3x}\frac{1}{D^2+6D+11}(x^2+1) + e^x\frac{1}{D^2+2D+3}\cos 2x \\ &= \frac{e^{3x}}{11}\frac{1}{\left(1+\frac{D^2+6D}{11}\right)}(x^2+1) + e^x\frac{1}{2D-1}\cos 2x \\ &= \frac{e^{3x}}{11}\left(1+\frac{D^2+6D}{11}\right)^{-1}(x^2+1) + e^x(2D+1)\frac{1}{4D^2-1}\cos 2x \\ &= \frac{e^{3x}}{11}\left(1-\frac{D^2+6D}{11}+\left(\frac{D^2+6D}{11}\right)^2-\cdots\right)(x^2+1) \\ &\qquad\qquad\qquad\qquad - \frac{1}{17}e^x(2D+1)\cos 2x \\ &= \frac{e^{3x}}{11}\left(x^2+1-\frac{2+12x}{11}+\frac{72}{121}\right) - \frac{1}{17}e^x(-4\sin 2x + \cos 2x) \\ &= \frac{e^{3x}}{11}\left(x^2-\frac{12x}{11}+\frac{171}{121}\right) - \frac{1}{17}e^x(-4\sin 2x+\cos 2x). \end{aligned}$$

Hence the general solution is

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} \\ &= C_1\cos(\sqrt{2}\,x) + C_2\sin(\sqrt{2}\,x) \\ &\qquad + \frac{e^{3x}}{11}\left(x^2-\frac{12x}{11}+\frac{171}{121}\right) - \frac{1}{17}e^x(-4\sin 2x+\cos 2x). \end{aligned}$$

□

Example 25. Solve $(D^2-2D+1)y = x^2e^{3x}$.

Rule for finding the particular integral when X is of the form xV, where V is a function of x.

Theorem 26. *In usual notation prove that* $\frac{1}{f(D)}xV = \left[x - \frac{1}{f(D)}f'(D)\right]\frac{1}{f(D)}V$, *where* V *is function of* x.

Proof. We have to evaluate $\frac{1}{f(D)}xV$. Let W be any function of x. Let $'$ denote the derivative with respect to D. Then

$$\begin{aligned} D(xW) &= xDW + W \\ &= xDW + \frac{d}{dD}(D)W \\ &= xDW + D'W. \end{aligned}$$

Also,

$$\begin{aligned} D^2(xW) &= D(xDW + W) \\ &= xD^2W + DW + DW \\ &= xD^2W + 2DW \\ &= xD^2W + \frac{d}{dD}(D^2)W \\ &= xD^2W + (D^2)'W. \end{aligned}$$

In general,

$$D^n(xW) = xD^nW + (D^n)'W.$$

Therefore,

$$\begin{aligned} f(D)xW &= (D^n + a_1D^{n-1} + \cdots + a_n)xW \\ &= xf(D)W + f'(D)W. \end{aligned}$$

Now suppose that W is given by $f(D)W = V$.
Hence, $W = \frac{1}{f(D)}V$.
Therefore, $f(D)x\frac{1}{f(D)}V = xV + f'(D)\frac{1}{f(D)}V$.
Operating by $\frac{1}{f(D)}$ on both the sides, we obtain,

$$\begin{aligned} x\frac{1}{f(D)}V &= \frac{1}{f(D)}xV + \frac{1}{f(D)}f'(D)\frac{1}{f(D)}V \\ \Rightarrow \frac{1}{f(D)}xV &= \left[x - \frac{1}{f(D)}f'(D)\right]\frac{1}{f(D)}V. \end{aligned}$$

Hence,

$$\frac{1}{f(D)}xV = \left[x - \frac{1}{f(D)}f'(D)\right]\frac{1}{f(D)}V,$$

where V is a function of x. □

Example 27. Solve $(D^2 + 9)y = x\sin x$.

Solution. Here the auxiliary equation is

$$D^2 + 9 = 0 \Rightarrow D = \pm 3i.$$

Thus the C.F. $= C_1\cos 3x + C_2\sin 3x$. Now,

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D^2+9}(x\sin x)\\
&= \left(x - \frac{1}{D^2+9}2D\right)\frac{1}{D^2+9}\sin x\\
&= x\frac{1}{D^2+9}\sin x - 2D\frac{1}{(D^2+9)^2}\sin x\\
&= x\frac{1}{8}\sin x - 2D\frac{1}{64}\sin x\\
&= \frac{x\sin x}{8} - \frac{\cos x}{32}.
\end{aligned}$$

Hence the general solution is

$$\begin{aligned}
y &= \text{C.F.} + \text{P.I.}\\
&= C_1\cos 3x + C_2\sin 3x + \frac{x\sin x}{8} - \frac{\cos x}{32}.
\end{aligned}$$

□

4. Solution of homogeneous linear differential equations

In this section, we consider the equation

$$x^n\frac{d^ny}{dx^n} + a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_ny = X, \tag{17}$$

which is called a *homogeneous linear differential equation of order n*, where $a_1, a_2, \ldots, a_n$ are constants and X is a function of x. This equation is also known as *Cauchy's homogeneous linear equation.*

In what follows we shall denote the derivative with respect to the variable z by D_1. To solve the equation (17), let $x = e^z$, that is, $z = \log x$. Hence $\frac{dz}{dx} = \frac{1}{x}$. Then $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}\frac{1}{x}$ or $x\frac{dy}{dx} = \frac{dy}{dz} = D_1y$, where $D_1 = \frac{d}{dz}$. Further,

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)\\
&= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)\\
&= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}\\
&= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}.
\end{aligned}$$

Hence,

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} = D_1^2y - D_1y = D_1(D_1-1)y,$$

where $D_1 = \frac{d}{dz}$. Similarly,

$$x^3\frac{d^3y}{dx^3} = (D_1^3 - 3D_1^2 + 2D_1)y = D_1(D_1-1)(D_1-2)y.$$

Continuing in this fashion we get,

$$x^n\frac{d^ny}{dx^n} = D_1(D_1-1)(D_1-2)\ldots(D_1-n+1)y.$$

Substituting all these values in (17) we have,

$$[D_1(D_1-1)\ldots(D_1-n+1) + a_1D_1(D_1-1)\ldots(D_1-n+2) + \cdots + a_n]y = Z.$$

That is,

$$\varphi(D_1)y = Z,$$

where Z is a function of z.

This is a linear differential equation with constant coefficients with z as an independent variable. This equation can now be solved by the methods already discussed.

Example 28. Solve

(1) $x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 15(x - x^{-1})$.

(2) $x^3\frac{d^2y}{dx^2} - 3x^2\frac{dy}{dx} + xy = \log x \cos(\log x)$.

Solution. (1) Here the given equation is a homogeneous linear equation. Let $x = e^z$, that is, $z = \log x$. Then the given equation becomes

$$(D_1(D_1-1)(D_1-2) + 2D_1(D_1-1) + 2)y = 15(e^z - e^{-z}),$$

$$\text{where} D_1 = \frac{d}{dz}$$
$$\Rightarrow (D_1^3 - D_1^2 + 2)y = 15(e^z - e^{-z}).$$

We solve this equation for y as a function of z. Here the auxiliary equation is

$$\begin{aligned}
&D_1^3 - D_1^2 + 2 = 0\\
\Rightarrow &D_1^2(D_1+1) - 2D_1(D_1+1) + 2(D_1+1) = 0\\
\Rightarrow &(D_1+1)(D_1^2 - 2D_1 + 2) = 0\\
\Rightarrow &D_1 = -1, 1 \pm i.
\end{aligned}$$

Thus C.F. $= C_1e^{-z} + e^z(C_2\cos z + C_3\sin z)$. That is,

$$\text{C.F.} = C_1x^{-1} + x(C_2\cos(\log x) + C_3\sin(\log x)).$$

Now,

$$\begin{aligned}
\text{P.I.} &= \frac{1}{D_1^3 - D_1^2 + 2}15(e^z - e^{-z})\\
&= 15\frac{1}{D_1^3 - D_1^2 + 2}e^z - 15\frac{1}{D_1^3 - D_1^2 + 2}e^{-z}\\
&= \frac{15}{2}e^z - 15\frac{1}{(D_1+1)(D_1^2 - 2D_1 + 2)}e^{-z}\\
&= \frac{15}{2}e^z - 15\frac{1}{5(D_1+1)}e^{-z}\\
&= \frac{15}{2}e^z - 3ze^{-z}\\
&= \frac{15}{2}x - 3x^{-1}\log x.
\end{aligned}$$

Hence the general solution is

$$\begin{aligned}
y &= \text{C.F.} + \text{P.I.}\\
&= C_1x^{-1} + x(C_2\cos(\log x) + C_3\sin(\log x)) + \frac{15}{2}x - 3x^{-1}\log x.
\end{aligned}$$

(2) Given equation can be written as

$$(x^2D^2 - 3xD + 1)y = x^{-1}\log x\cos(\log x), \tag{18}$$

which is a homogeneous linear equation. Let $x = e^z$, that is, $z = \log x$. Then equation (18) becomes

$$\begin{aligned}
&(D_1(D_1-1) - 3D_1 + 1)y = e^{-z}z\cos z\\
\Rightarrow\ &(D_1^2 - 4D_1 + 1)y = e^{-z}z\cos z.
\end{aligned}$$

Hence the auxiliary equation is

$$D_1^2 - 4D_1 + 1 = 0$$
$$\Rightarrow D_1 = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}.$$

$$\begin{aligned}\text{Hence, C.F. } &= C_1 e^{(2+\sqrt{3})z} + C_2 e^{(2-\sqrt{3})z} \\ &= C_1 e^{(2+\sqrt{3})\log x} + C_2 e^{(2-\sqrt{3})\log x} \\ &= C_1 x^{2+\sqrt{3}} + C_2 x^{2-\sqrt{3}} \\ &= x^2(C_1 x^{\sqrt{3}} + C_2 x^{-\sqrt{3}}).\end{aligned}$$

Now,

$$\begin{aligned}\text{P.I. } &= \frac{1}{D_1^2 - 4D_1 + 1} e^{-z}(z\cos z) \\
&= e^{-z}\frac{1}{(D_1-1)^2 - 4(D_1-1) + 1}(z\cos z) \\
&= e^{-z}\frac{1}{D_1^2 - 6D_1 + 6}(z\cos z) \\
&= e^{-z}\left[z\frac{1}{D_1^2 - 6D_1 + 6}\cos z - \frac{1}{(D_1^2 - 6D_1 + 6)^2}(2D_1 - 6)\cos z\right] \\
&= e^{-z}\left[z\frac{1}{D_1^2 - 6D_1 + 6}\cos z - \frac{1}{(D_1^2 - 6D_1 + 6)^2}(-2\sin z - 6\cos z)\right] \\
&= e^{-z}\left[z\frac{1}{5 - 6D_1}\cos z + \frac{1}{(5 - 6D_1)^2}(2\sin z + 6\cos z)\right] \\
&= e^{-z}\left[z(5+6D_1)\frac{1}{25 - 36D_1^2}\cos z + \frac{1}{25 - 60D_1 + 36D_1^2}(2\sin z + 6\cos z)\right] \\
&= e^{-z}\left[z(5+6D_1)\frac{1}{61}\cos z + \frac{1}{-11 - 60D_1}(2\sin z + 6\cos z)\right] \\
&= e^{-z}\left[\frac{z}{61}(5\cos z - 6\sin z) - (60D_1 - 11)\frac{1}{3600D_1^2 - 121}(2\sin z + 6\cos z)\right] \\
&= e^{-z}\left[\frac{z}{61}(5\cos z - 6\sin z) + (60D_1 - 11)\frac{1}{3721}(2\sin z + 6\cos z)\right] \\
&= e^{-z}\left[\frac{z}{61}(5\cos z - 6\sin z) + \frac{1}{3721}(120\cos z - 360\sin z - 22\sin z - 66\cos z)\right] \\
&= e^{-z}\left[\frac{z}{61}(5\cos z - 6\sin z) + \frac{1}{3721}(54\cos z - 382\sin z)\right] \\
&= x^{-1}\left[\frac{\log x}{61}(5\cos(\log x) - 6\sin(\log x)) + \frac{1}{3721}(54\cos(\log x) - 382\sin(\log x))\right].\end{aligned}$$

Hence the general solution is $y = \text{C.F.} + \text{P.I.}$

$$y = x^2(C_1 x^{\sqrt{3}} + C_2 x^{-\sqrt{3}}) + x^{-1}\left[\tfrac{\log x}{61}(5\cos(\log x) - 6\sin(\log x)) + \tfrac{1}{3721}(54\cos(\log x) - 382\sin(\log x))\right].$$

□

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